2.2.4 Lipschitz Continuity of Convex Functions

Our goal in this section is to show that convex functions are Lipschitz continuous inside the interior of its domain.

We will first show that a convex function is locally bounded.

Lemma 2.2. *Let f be convex and $x_0 \in \operatorname{int}\operatorname{dom} f$. Then f is locally bounded, i.e., $\exists \varepsilon > 0$ and $M(x_0, \varepsilon) > 0$ such that*

$$f(x) \le M(x_0, \varepsilon) \;\forall\, x \in B_\varepsilon(x_0) := \{x \in \mathbb{R}^n : \|x - x_0\|_2 \le \varepsilon\}.$$

Proof. Since $x_0 \in \operatorname{int}\operatorname{dom} f$, $\exists \varepsilon > 0$ such that the vectors $x_0 \pm \varepsilon e_i \in \operatorname{int}\operatorname{dom} f$ for $i = 1, \ldots, n$, where e_i denotes the unit vector along coordinate i. Also let $H_\varepsilon(x_0) := \{x \in \mathbb{R}^n : \|x - x_0\|_\infty \le \varepsilon\}$ denote the hypercube formed by the vectors $x_0 \pm \varepsilon e_i$. It can be easily seen that $B_\varepsilon(x_0) \subseteq H_\varepsilon(x_0)$ and hence that

$$\max_{x \in B_\varepsilon(x_0)} f(x) \le \max_{x \in H_\varepsilon(x_0)} f(x) \le \max_{i=1,\ldots,n} f(x_0 \pm \varepsilon e_i) =: M(x_0, \varepsilon).$$

■

Next we show that f is locally Lipschitz continuous.

Lemma 2.3. *Let f be convex and $x_0 \in \operatorname{int}\operatorname{dom} f$. Then f is locally Lipschitz, i.e., $\exists \varepsilon > 0$ and $\bar{M}(x_0, \varepsilon) > 0$ such that*

$$|f(y) - f(x_0)| \le \bar{M}(x_0, \varepsilon)\|x - y\|, \;\forall y \in B_\varepsilon(x_0) := \{x \in \mathbb{R}^n : \|x - x_0\|_2 \le \varepsilon\}. \tag{2.2.10}$$

Proof. We assume that $y \neq x_0$ (otherwise, the result is obvious). Let $\alpha = \|y - x_0\|_2/\varepsilon$. We extend the line segment connecting x_0 and y so that it intersects the ball $B_\varepsilon(x_0)$, and then obtain two intersection points z and u (see Fig. 2.4). It can be easily seen that

$$y = (1 - \alpha)x_0 + \alpha z, \tag{2.2.11}$$
$$x_0 = [y + \alpha u]/(1 + \alpha). \tag{2.2.12}$$

It then follows from the convexity of f and (2.2.11) that

$$\begin{aligned} f(y) - f(x_0) &\le \alpha[f(z) - f(x_0)] = \tfrac{f(z) - f(x_0)}{\varepsilon}\|y - x_0\|_2 \\ &\le \tfrac{M(x_0,\varepsilon) - f(x_0)}{\varepsilon}\|y - x_0\|_2, \end{aligned}$$

where the last inequality follows from Lemma 2.2. Similarly, by the convexity f, (2.2.11), and Lemma 2.2, we have

$$f(x_0) - f(y) \le \|y - x_0\|_2 \tfrac{M(x_0,\varepsilon) - f(x_0)}{\varepsilon}.$$

Combining the previous two inequalities, we show (2.2.10) holds with $\bar{M}(x_0, \varepsilon) = [M(x_0, \varepsilon) - f(x_0)]/\varepsilon$. ■

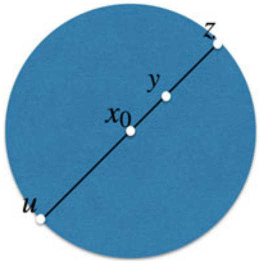

Fig. 2.4 Local Lipschitz continuity of a convex function

The following simple result shows the relation between the Lipschitz continuity of f and the boundedness of subgradients.

Lemma 2.4. *The following statements hold for a convex function f.*

(a) If $x_0 \in \mathrm{int\,dom} f$ and f is locally Lipschitz (i.e., (2.2.10) holds), then $\|g(x_0)\| \le \bar{M}(x_0,\varepsilon)$ for any $g(x_0) \in \partial f(x_0)$.
(b) If $\exists g(x_0) \in \partial f(x_0)$ and $\|g(x_0)\|_2 \le \bar{M}(x_0,\varepsilon)$, then $f(x_0) - f(y) \le \bar{M}(x_0,\varepsilon)\|x_0 - y\|_2$.

Proof. We first show part (a). Let $y = x_0 + \varepsilon g(x_0)/\|g(x_0)\|_2$. By the convexity of f and (2.2.10), we have

$$\varepsilon\|g(x_0)\|_2 = \langle g(x_0), y - x_0\rangle \le f(y) - f(x_0) \le \bar{M}(x_0,\varepsilon)\|y - x_0\| = \varepsilon\bar{M}(x_0,\varepsilon),$$

which implies part (a). Part (b) simply follows the convexity of f, i.e.,

$$f(x_0) - f(y) \le \langle g(x_0), x_0 - y\rangle \le \bar{M}(x_0,\varepsilon)\|x_0 - y\|_2.$$

■

Below we state the global Lipschitz continuity of a convex function in its interior of domain.

Theorem 2.4. *Let f be a convex function and let K be a closed and bounded set contained in the relative interior of the domain* $\mathrm{dom} f$ *of f. Then f is Lipschitz continuous on K, i.e., there exists constant M such that*

$$|f(x) - f(y)| \le M_K\|x - y\|_2 \quad \forall x, y \in K. \tag{2.2.13}$$

Proof. The result directly follows from the local Lipschitz continuity of a convex function (see Lemmas 2.3 and 2.4) and the boundedness of K. ■

Remark 2.1. All three assumptions on K—i.e., (a) closedness, (b) boundedness, and (c) $K \subset \mathrm{ridom} f$—are essential, as it is seen from the following three examples:

- $f(x) = 1/x$, $\mathrm{dom} f = (0, +\infty)$, $K = (0, 1]$. We have (b), (c) but not (a); f is neither bounded, nor Lipschitz continuous on K.
- $f(x) = x^2$, $\mathrm{dom} f = \mathbb{R}$, $K = \mathbb{R}$. We have (a), (c) and not (b); f is neither bounded nor Lipschitz continuous on K.
- $f(x) = -\sqrt{x}$, $\mathrm{dom} f = [0, +\infty)$, $K = [0, 1]$. We have (a), (b) and not (c); f is not Lipschitz continuous on K although is bounded. Indeed, we have $\lim_{t\to+0} \frac{f(0)-f(t)}{t} = \lim_{t\to+0} t^{-1/2} = +\infty$, while for a Lipschitz continuous f the ratios $t^{-1}(f(0) - f(t))$ should be bounded.

2.2.5 Optimality Conditions for Convex Optimization

The following results state the basic optimality conditions for convex optimization.

Proposition 2.6. *Let f be convex. If x is a local minimum of f, then x is a global minimum of f. Furthermore this happens if and only if $0 \in \partial f(x)$.*

Proof. It can be easily seen that $0 \in \partial f(x)$ if and only if x is a global minimum of f. Now assume that x is a local minimum of f. Then for $\lambda > 0$ small enough one has for any y,

$$f(x) \le f((1-\lambda)x + \lambda y) \le (1-\lambda) f(x) + \lambda f(y),$$

which implies that $f(x) \le f(y)$ and thus that x is a global minimum of f. ■

The above result can be easily generalized to the constrained case. Given a convex set $X \subseteq \mathbb{R}^n$ and a convex function $f : X \to \mathbb{R}$, we intend to

$$\min_{x\in X} f(x).$$

We first define the indicator function of the convex set X, i.e.,

$$I_X(x) := \begin{cases} 0, & x \in X, \\ \infty, & Otherwise. \end{cases}$$

By definition of subgradients, we can see that the subdifferential of I_X is given by the normal cone of X, i.e.,

$$\partial I_X(x) = \{w \in \mathbb{R}^n | \langle w, y-x \rangle \le 0, \forall y \in X\}. \tag{2.2.14}$$

Proposition 2.7. *Let $f : X \to \mathbb{R}$ be a convex function and X be a convex set. Then x^* is an optimal solution of $\min_{x\in X} f(x)$ if and only if there exists $g^* \in \partial f(x^*)$ such that*

$$\langle g^*, y - x^* \rangle \ge 0, \forall y \in X.$$